End of 19^{th}, beginning of 20^{th} century:

E&M was well established

Biggest problems in EM theory:

1. light is a wave of oscillating $\vec{E}$ & $\vec{B}$ fields
   $\Rightarrow$ all known oscillations involve propagation of a disturbance

   e.g. water waves: waves travel along surface of water - water level is "disturbed" $\rightarrow$ no water, no waves

   e.g. sound: pressure disturbance in air $\longrightarrow$ no air, no sound

   so what is being disturbed in the case of light?

   $\Rightarrow$ guess: "ether" - invisible "fabric"

note: the "stiffer" the medium, the faster the propagation

ex: speed of sound in air: ~340 m/s
lead: 1210 m/s
aluminum: 6320 m/s
speed of light in "ether": 300,000,000 m/s

since $c = 3\times10^8$ m/s, ether should be extremely stiff! but it's invisible?

2. wave equation for EM waves in a vacuum w/no sources (starting w/ Maxwell's equations)

$$\frac{\partial^2 E}{\partial x^2} - \varepsilon_0 \mu_0 \frac{\partial^2 E}{\partial t^2} = 0$$

this equation describes a propagating wave along x-direction:

$$E = E_0 \cos(kx - \omega t) \quad \text{where } \frac{\omega}{k} = \frac{1}{\sqrt{\varepsilon_0 \mu_0}} = v \equiv c$$

- This says c = constant which implies that there is some absolute reference frame where $v = 3\times10^8$ m/s for EM radiation
- Then if you are moving w/some velocity relative to the ether you can measure your velocity in the ether by measuring speed of light in different directions

$\Rightarrow$ if ether exists then it would be the reference frame of absolute rest **OR**

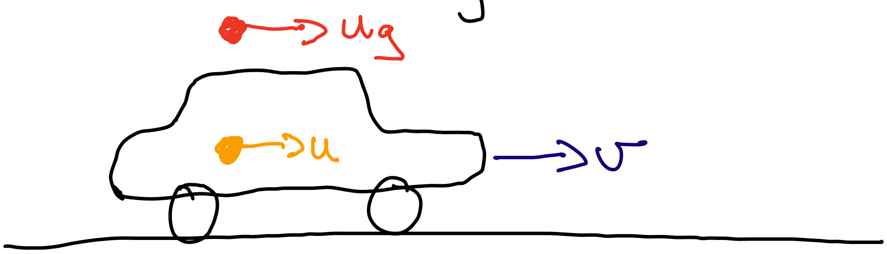
$\Rightarrow$ if there was an absolute ref frame then that means there's an ether

AND the laws of nature would depend on your velocity

Reference frame: the frame where you are <u>not</u> moving (you are standing still)

ex: you drive in a car. The car is your reference frame. You pass me, I'm in the reference frame of the ground

let car have velocity v <u>relative to ground</u>



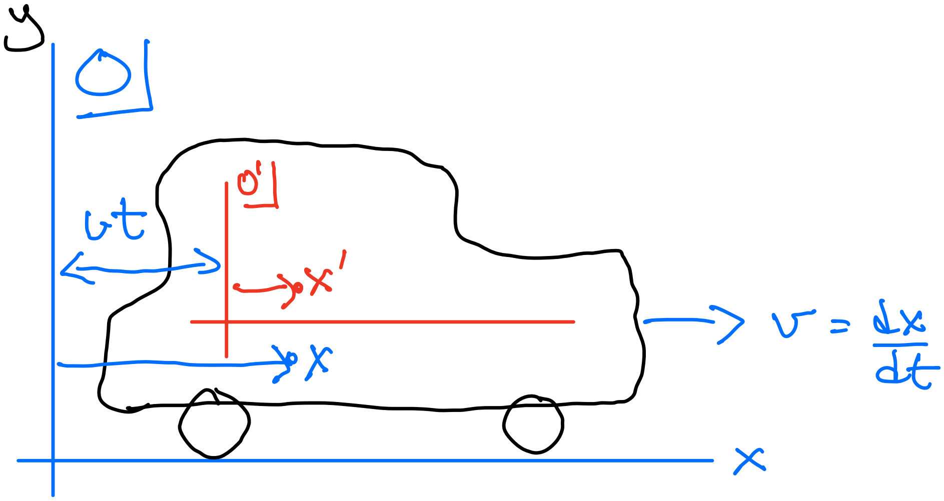
in the car someone throws a ball from back to front seat w/ velocity u

$\Rightarrow$ what is ball's velocity (u_g) relative to ground?

The answer is easy: $u_g = v + u$

More rigorous derivation: coordinate systems

Let O be reference frame of ground
and O' " " " in the car



$v = \frac{dx}{dt}$ where "x" is any coordinate position of the car in frame O

x' = coordinate position inside the car (relative to some agreed upon point in the car)

then x on ground is related to x' in the car by how far the car has moved in time t

$x = x' + vt$ } "galilean" transformation now calculate $u_g = \frac{dx}{dt}$ velocity of ball relative to ground

$$u_g = \frac{dx}{dt} = \frac{d}{dt}(x' + vt) = \underbrace{\frac{dx'}{dt}}_{\substack{\text{how position of ball} \\ \text{in car frame } O' \text{ is changing}}} + v = u + v$$

Problem w/ EM waves: if you shine light in the car forward and I measure its velocity on the ground, Galileo says I should get

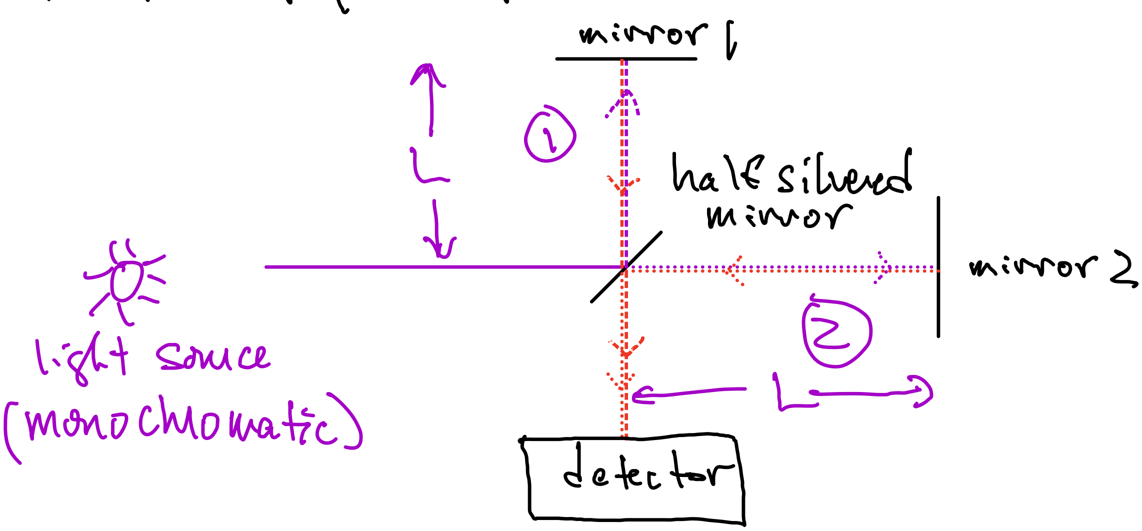
$$\underset{\substack{\text{speed of} \\ \text{light in} \\ \text{ground's frame}}}{u_{light}} = \underset{\substack{\text{speed of} \\ \text{light in car} \\ \text{frame}}}{c} + \underset{\substack{\text{speed of car} \\ \text{in ground's frame}}}{v}$$

so $u_{light} > c$ violates Maxwell's eqns

So to measure your velocity relative to the ether, you are measuring the ~~absolute~~ velocity inside the universe where $v=0$ absolute (ether!)

Michelson-Morley experiment
1887, Case Western university, Cleveland

Built interferometer:



- light from source hits a "half mirror" and splits beam into 2 parts

path 1. half light reflected up, then reflects back down by mirror 1 (dashes).

It then hits half mirror again and half goes thru to detector (ignore other half)

path 2. other half goes thru half mirror and is reflected back by mirror 2 (dotted).
It then hits half mirror again & half gets reflected down to detector

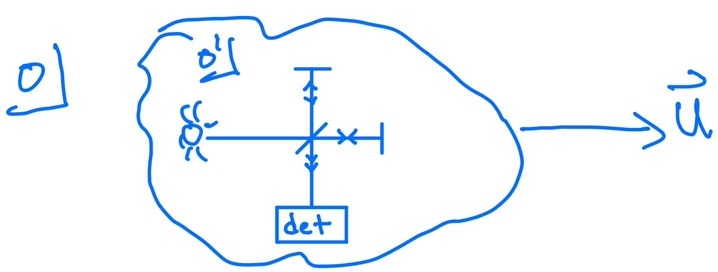
Make dist from half mirror to the 2 mirrors the same: L

Both beams interfere w/each other at detector.

If there were no ether then you could adjust the distances so that the 2 waves have a path difference $\Delta r = n\lambda$ where n is some number.

The waves would interfere constructively & you would see a bright spot.

Next say you were moving w/some velocity relative to the ether along horizontal.



$x = x' + ut$

or $\Delta x = \Delta x' + u\Delta t$

t_1 = time for horizontal beam to go from half mirror to full mirror

t_2 = time from full mirror back to half

$t_1 > t_2$ because mirror is moving in the same direction as light

remember light moves at $v = c$ in ether only!!

Horizontal beam

- beam travels with vel = u in dir parallel to earth moving thru ether
- horizontal beam goes dist L in earth frame in time t_1

dist in ether is $c\Delta t_1 = L + u\Delta t_1$

on return from mirror to half splitter:

$$c\Delta t_2 = L - u\Delta t_2$$

light is anti parallel to $\vec{u}$

$\Delta t_1 > \Delta t_2$ because on 1st leg, destination (mirror) is moving away from starting pt (half mirror)

on return trip destination is moving towards starting point calculate total time in ether frame:

$$c\Delta t_1 = L + u\Delta t_1$$

$$(c-u)\Delta t_1 = L$$

$$\Delta t_1 = \frac{L}{c-u} = \frac{L}{c}\frac{1}{1-\beta} \qquad \beta \equiv u/c$$

$$\Delta t_2 = \frac{L}{c+u} = \frac{L}{c}\frac{1}{1+\beta}$$

here you see $\Delta t_1 > \Delta t_2$

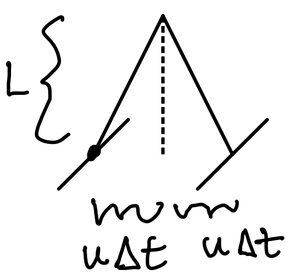
total time is $\Delta t_1 + \Delta t_2 = \frac{L}{c}\left(\frac{1}{1-\beta} + \frac{1}{1+\beta}\right)$

$$= \frac{L}{c}\left(\frac{1+\beta+1-\beta}{1-\beta^2}\right) = \frac{2L}{c}\frac{1}{1-\beta^2} \qquad \text{let } \gamma^2 = \frac{1}{1-\beta^2}$$

$\Delta t_h = \frac{2L\gamma^2}{c}$ "horizontal" time Δt_h

Vertical beam

This is the beam that is $\perp$ earth's motion thru ether



total distance is $2\sqrt{L^2+(u\Delta t_v)^2}$

this has to be $2c\Delta t$ in ether frame

so $2c\Delta t_v = 2\sqrt{L^2+(u\Delta t_v)^2}$

$$c^2 \Delta t_v^2 = L^2 + u^2 \Delta t_v^2$$
$$L^2 = (c^2 - u^2) \Delta t_v^2$$
$$\Delta t_v = \frac{L}{\sqrt{c^2 - u^2}} = \frac{L}{c} \frac{1}{\sqrt{1-\beta^2}} = \frac{L\gamma}{c}$$

total time $\Delta t_v = 2\Delta t = \frac{2L\gamma}{c}$

and $\Delta t_h = \frac{2L\gamma^2}{c}$

so $\Delta t_h > \Delta t_v$

let's say $\beta \ll 1$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = (1-\beta^2)^{-1/2} \sim \left(1 + \frac{1}{2}\beta^2\right)$$ expand and keep smallest terms

$$\gamma^2 = \frac{1}{1-\beta^2} = (1-\beta^2)^{-1} \sim 1 + \beta^2$$

$$\Delta t \equiv \Delta t_h - \Delta t_v = \frac{2L\gamma^2}{c} - \frac{2L\gamma}{c}$$
$$\sim \frac{2L}{c}\left(1 + \beta^2 - \left(1 + \frac{1}{2}\beta^2\right)\right) \sim \frac{2L}{c}\frac{\beta^2}{2}$$
$$= \beta^2 \frac{L}{c}$$

in that time, horizontal light goes dist

$$d = c\Delta t = \beta^2 L$$

light has wavelength ~ 500nm

if $d = \beta^2 L \cong 50\text{nm}$, you would see it in the interference pattern changing

for $L = 10\text{m}$ interferometer arm:

$d = 50\times10^{-9}\text{m} = \beta^2 \cdot 10\text{m}$

$$\beta^2 = 50\times10^{-10}$$

$\beta = \frac{u}{c} \Rightarrow \quad u = c * \sqrt{50\times10^{-10}} \cong 3\times10^8 \frac{m}{s} \cdot 7\times10^{-5}$

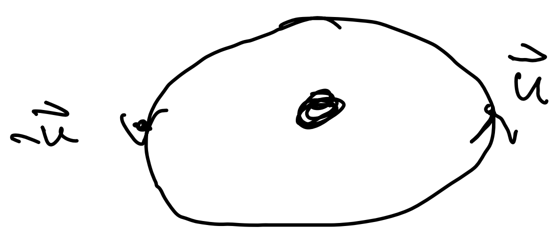
$$= 21\times10^3 \, m/s$$

$$\sim 2\times10^4 \, m/s$$

1. take interferometer & measure interference pattern, then adjust one full mirror distance so that you see a max

2. turn it at right angle - should see a big change

3. do the same 6 months later when on opposite side of sun



200 3/26

No effect seen!!!

No experiment ever tried has ever seen any evidence of ether.

So if the ether doesn't exist, then

(Einstein's POSTULATES)

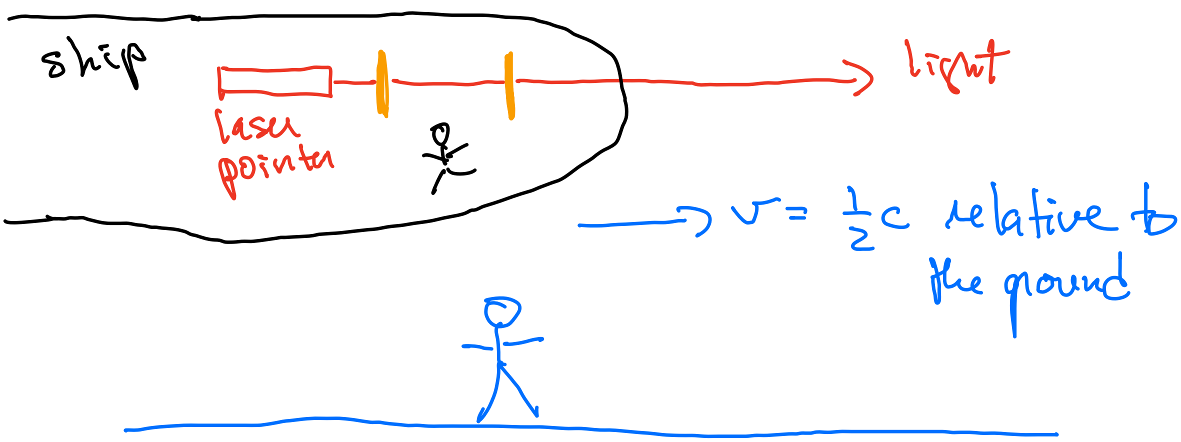
1. There is no absolute ref frame. So no absolute velocities

   Conclusion: the laws of nature only depends on relative velocities

   more accurate: laws of nature are the same in all ref frames that have constant v ("inertial frames")

2. Maxwell's equations say $v_{light} = 3\times10^8\,m/s$

   ⟹ but in which ref frame? all frames? this violates common sense!



- ship has a laser shining light along direction of motion.

  person on ship can measure light velocity by timing beam as it passes thru the two detectors.

  ⇒ will measure $v_{light} = c$ on ship

- person on ground can also measure light velocity w/ duplicate detectors on the ground

  will measure $v_{light} = c$ on ground!

  and <u>not</u> $v_{light} = c + v_{ship}$

The 2 postulates above are "Einstein's Postulates"

#2 says that the speed of light is the largest possible <u>relative</u> velocity

How can both postulates be true <u>and</u> the Galilean transformation hold?

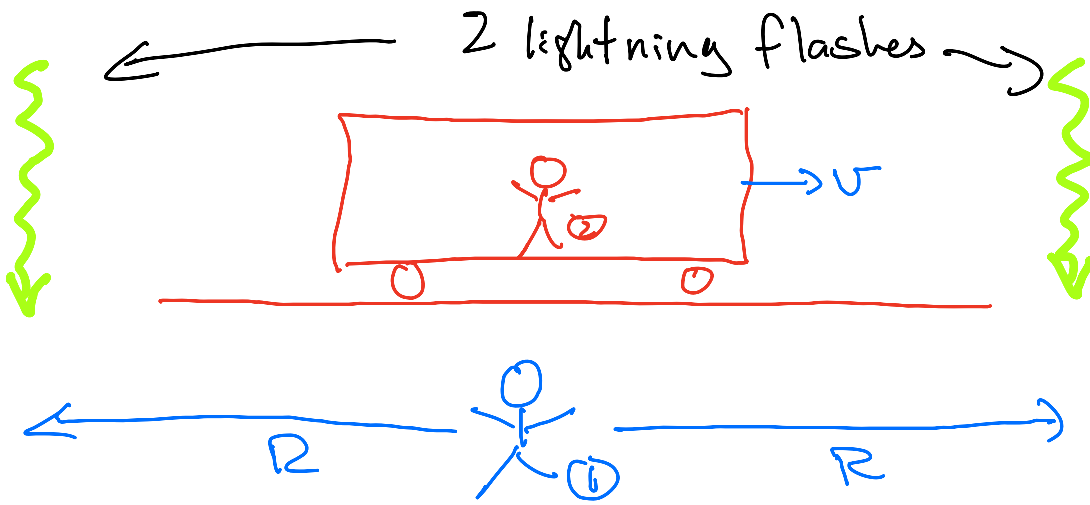
Galilean: $x = x' + vt$ and time is absolute

⇒ figuring this out is one of Einstein's many contributions

400 3/26

Simutaneity & time

The beginning....



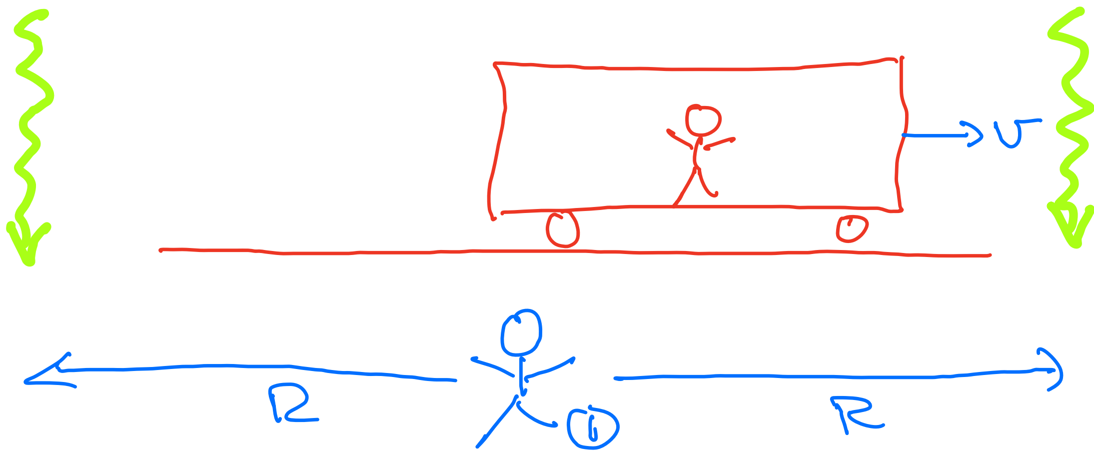
person ① on ground, person ② on car w/vel. = v moving to right.

lightening strikes the ground the same distance R to left and right of person ①

① says lightning bolts were simulttaneous because the light arrived at ① at the same time

② lightning flash took t_1 seconds to get to person ①

③ person ② goes a distance $d_2 = v t_1$ in time t_1 and then sees 1st flash



2 sees the flash from the bolt on right 1^{st}, then the flash from the left next

$\Rightarrow$ 1 & 2 disagree that the 2 events (each flash is an event) were simultaneous (at the same time)

But – postulate I says all inertial reference frames are equivalent

Therefore: simultaneity must not be anything fundamental

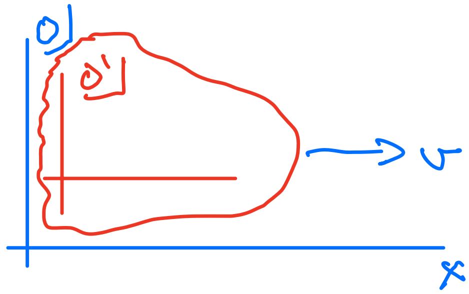
Therefore: time is <u>not</u> absolute!

$\Rightarrow$ implies that we have to modify Galilean transformation $\Rightarrow$ time is relative!

But How???

O| is our ref frame

O'| moves w/rel velocity v in O|



Galilean: $x = x' + vt$ needs adjustments!

1. try $x = f(v)(x' + vt)$

   where $f(v) \longrightarrow 1$ as $v \to 0$

   since Galilean does seem to work

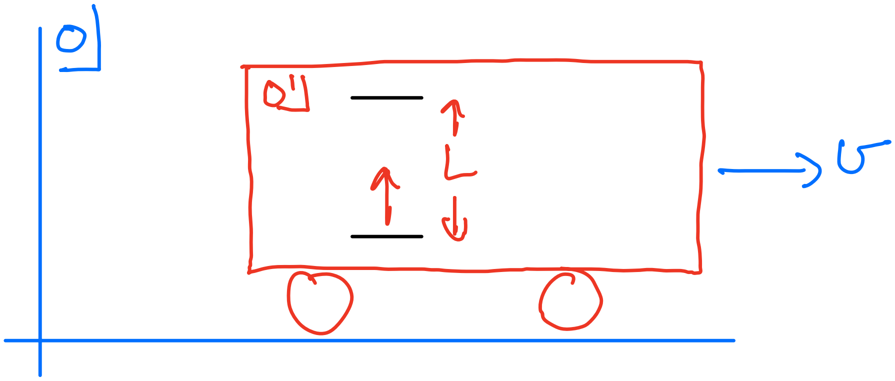
   when $v \ll c$ (the "real" world)

2. since time is relative and not absolute, then time could be different in the 2 frames O| & O'|

   so $x = f(v)(x' + vt')$ $t \neq t'$

new questions:

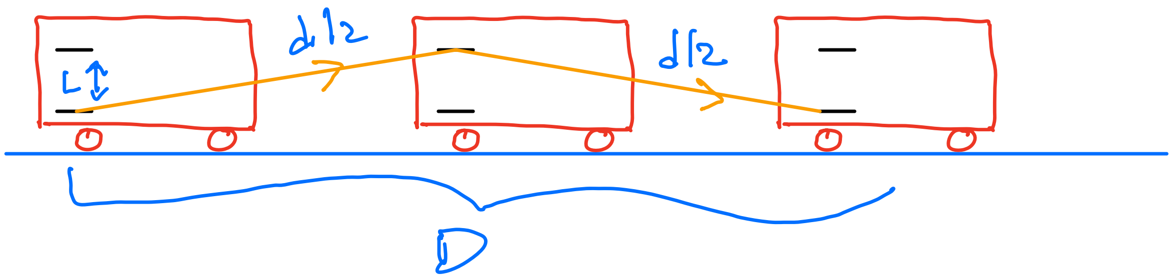
- how to calculate $f(v)$?
- how does time "transform"?



- 2 mirrors on train moving w/ velocity v in O]
- train is O'] frame
- bounce a beam of light between the mirrors

  $\Rightarrow$ takes $\Delta t'$ time to go distance $2L$ in train, velocity of light is c

  $\therefore$ $2L = c\Delta t'$ (t' is time in O'])
- in frame O], we see this



Distance train travels in time Δt: $D = v\Delta t$

Distance light travels along diagonal: $d = c\Delta t$

Pythagorus: $\left(\frac{d}{2}\right)^2 = \left(\frac{D}{2}\right)^2 + L^2$

and $d = c\Delta t$ since light travels at vel c in ref frame O in all frames

so $\left(\frac{d}{2}\right)^2 = \left(\frac{D}{2}\right)^2 + L^2$ use $L = \frac{c\Delta t'}{2}$ from O'

$$\left(\frac{c\Delta t}{2}\right)^2 = \left(\frac{v\Delta t}{2}\right)^2 + \left(\frac{c\Delta t'}{2}\right)^2$$

rearrange, get rid of "2":

$$c^2\Delta t^2 - v^2\Delta t^2 = c^2\Delta t'^2 \Rightarrow \Delta t' = \Delta t\sqrt{1-\underbrace{\frac{v^2}{c^2}}_{\beta^2}}$$

define $\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$ $\Rightarrow \Delta t = \Delta t'\gamma$

200 3/27

note: O' is called the "proper frame" and

$\Delta t'$ is the "proper time" $\equiv \Delta\tau$

Proper frame: frame where events are happening at locations (x') that are <u>not</u> changing. $\Rightarrow$ <u>your</u> reference frame

Proper time is the time in the frame where position isn't changing

ex: you are on an airplane and hold your breath for 60 sec.

in your frame, position isn't changing
so proper time $\Delta\tau = 60$ sec

then a time interval in some frame moving w/ velocity v relative to proper frame is Δt and

$$\boxed{\Delta t = \gamma \Delta\tau} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}}, \ \beta \equiv v/c$$

$\Delta\tau$ is always the shortest time interval than the interval in any other frame

This is called $\boxed{\text{"time dilation"}}$

ex: you are moving w/ velocity v relative to me. a year goes by in your frame

$$\Delta\tau = 1 \text{ year}$$

what do I measure?

$$\Delta t = \gamma \Delta\tau$$

if $v = 10{,}000$ mph:

$$v = 10^4 \frac{mi}{hr} * \frac{5280ft}{mi} * \frac{1\,m}{3.28ft} * \frac{1\,hr}{3600s}$$

$$= 10^4\, mph * 0.45 \frac{m/s}{mph} = 4500\, m/s$$

$(1\, mph = 0.45\, m/s)$

$$\beta = \frac{v}{c} = \frac{4500}{3\times10^8} = 1.5\times10^{-5}$$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = (1-\beta^2)^{-1/2} \approx 1 + \underbrace{\tfrac{1}{2}\beta^2}_{10^{-10}!} \sim 1$$ **no effect**

what if $\beta = 0.1$? $(v = 3\times10^7\, m/s = 18{,}600\, mi/s!)$

$$\gamma = \frac{1}{\sqrt{1-0.1^2}} = (1-0.1^2)^{-1/2} \sim 1 + \tfrac{1}{2}(0.1)^2 = 1 + 0.005 = 1.005$$

$$\Delta t = 1.005\, \Delta\tau = 1.005\, yr$$

$$0.005\, yr * \frac{365d}{yr} = 1.825\ \text{days}!$$

ex: a muon is a particle that decays on average after $2.2\mu s$

muons are made constantly in the upper atmosphere when cosmic rays hit the atmosphere. They are created with velocities

$\beta = 0.99$ wrt earth frame

How long do the muons live in earth's frame?

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-0.99^2}} = 7.09$$

$\Delta\tau = 2.2\mu s$ in muon's rest frame (where it is "standing still")

$$\Delta t_{earth} = \gamma\Delta\tau = 7.09 \times 2.2\mu s = 15.6\mu s$$

in $15.6\mu s$ muons travel distance (in earth's frame)

$$D = vt = .99c * 15.6\mu s$$
$$= 0.99 \times 3 \times 10^8 \frac{m}{s} \times 15.6 \times 10^{-6} s$$
$$= 4631 m = 4.6 km \sim 3 \text{ miles}$$

200 4/2

"Twin paradox"

Twins Alpha & Beta, same age
Alpha stays on earth.
Beta flies away on spaceship, $\beta = 0.6$ towards nearest star α-Centauri, 5 lt-yrs away

Alpha :
Beta :

$$\beta = 0.6 \quad \text{so} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-0.6^2}} = 1.25$$

Alpha frame (earth): Beta goes 5 lt-yrs at $\beta = 0.6$

$d = v \cdot \Delta t_\alpha = \frac{v}{c} * (\Delta t_\alpha \cdot c) = \beta c \Delta t_\alpha$

note: $c \cdot \Delta t$ is distance light travels in time Δt

$\frac{v}{c} = \beta$ dimensionless number

so if we measure distances in light-time (e.g. light-years) and time in years then $v \equiv \beta$

so $v = 0.6$ and $d =$ light-years so $\Delta t_\alpha =$ years

$\Delta t_\alpha = \frac{d}{v} = \frac{5 \text{ lt-years}}{0.6\, c} = 8.3$ years time interval in Alpha frame

Alpha measure's Beta clock time: proper time in Beta frame

$\Delta t_\alpha = \gamma \Delta t_\beta$

$\therefore \Delta t_\beta = \frac{\Delta t_\alpha}{\gamma} = \frac{8.3\text{yr}}{1.25} = 6.7\text{yrs}$

Beta turns around and heads home at same velocity

total time in Alpha (earth) frame:

$\Delta t_\alpha = 2 \times 8.3\text{yrs} = 16.6\text{ yrs}$

total time in Beta (ship) frame:

$\Delta t_\beta = 2 \times 6.7\text{yrs} = 13.4\text{ yrs}$

Alpha is now older than Beta!

⇒ Relativity says you can time travel into the future!

Huh? Paradox since shouldn't it be relative?

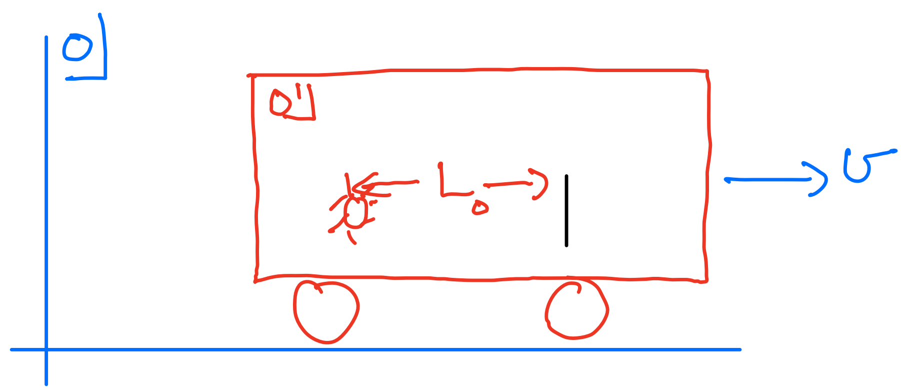
NO! Beta had to accelerate from $v = 0$ on earth to $v = 0.6c$, then decelerate to stop & turn around, then reaccelerate

⇒ Alpha is in inertial frame but Beta is not

400 4/2

Time dilation covers time intervals

⇒ what about length intervals



shine a light in O' (proper frame), it hits a mirror a distance L away and bounces back so $c\Delta t' = 2L_0$ $\Delta t'$ is transit time in O'

in frame O) the time to go from source to mirror is measured to be Δt_1 and the distance is L which maybe is not the same as L_0!

in O total distance light traveled to mirror

$$d_1 = c\Delta t_1 = L + v\Delta t_1$$ (>L since O' moving in same dir as light)

so $L = (c-v)\Delta t_1$

or $\Delta t_1 = \dfrac{L}{c-v}$

on return trip, mirror to source, measured Δt_2

and $d_2 = c\Delta t_2 = L - v\Delta t_2$ (<L since O' moving in opposite dir as light)

so $\Delta t_2 = \dfrac{L}{c+v}$

then $\Delta t = \Delta t_1 + \Delta t_2 = \dfrac{L}{c-v} + \dfrac{L}{c+v}$

$$= \frac{L(c+v+c-v)}{(c-v)(c+v)} = \frac{2Lc}{c^2-v^2}$$

$$\Delta t = \frac{2L}{c}\frac{1}{1-v^2/c^2} = \frac{2L}{c}\gamma^2$$

also: $c\Delta t' = 2L_0$ (proper frame) and we know that $\Delta t = \Delta t' \gamma$ ($\Delta t'$ = proper time)

so $\frac{2L\gamma^2}{c} = \left(\frac{2L_0}{c}\right)\gamma$

$$\boxed{L = \frac{L_0}{\gamma}}$$

$L < L_0$! (L_0 = length in proper frame)

This is called "Lorentz contraction" or "length " "

200 4/4

ex: a spaceship goes at $\beta = 0.9$ and is 100m long. This means in proper frame of the spaceship, it is measured to be 100m ($L_0 = 100m$)

what is the length on earth's frame where it goes at $\beta = 0.9$?

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-0.9^2}} = 2.3$$

on earth $L = \frac{L_0}{\gamma} = \frac{100}{2.3} = 43.6m$!

how can this be?

It's because of the relativity of simultaneity. On earth as the ship passes your 100m ruler, you record the position of the front and back of the ship AT THE SAME TIME !!

But on the ship, they will say you measured the front 1st, then the back.

But this means that you could have the 100m ship in a room with front & rear doors closed at same time (in your frame) and it will fit!

How to use this?

Galilean: $x = x' + vt'$, $t = t'$ } this is <u>very</u> accurate!

Relativistic transformation has to reduce to Galilean when $v \to 0$

try $x = f(v)(x' + vt')$
and $f(v) = f(-v)$ and
$f(0) = 1$

if you reverse frames, then O goes at vel $-v$ with respect to O'

and $x' = x - vt$

so try $x' = f(v)(x - vt)$ relativistic

This also works for intervals:

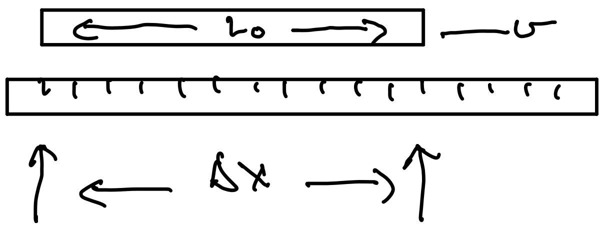
$$\Delta x' = f(v)(\Delta x - v\Delta t)$$

Now we measure the length of a moving object:

it has length Lo in the proper frame O'

$$\Delta x' = L_0$$

in O, we measure the length by recording the beginning & ends of the moving object with a stationary ruler



we get a length $L = \Delta x$

when we record the ruler values, we do so simultaneously at both ends

so $\Delta t = 0$ between the events (recording)

transforming back:

$$\Delta x' = f(v)(\Delta x - v\Delta t)$$

but $\Delta t = 0$

Lorentz contraction

so $\Delta x' = f(v)\Delta x$

$$L_0 = f(v) L = f(v)\frac{L_0}{\gamma}$$

so $f(v) = \gamma$

so the coordinate transformations are:

$$x = \gamma(x' + vt') = \gamma(x' + \beta c t')$$

$x' = \gamma(x - \beta c t)$
$y' = y$ (perpendicular to $\vec{v}$)

These are relativistic transformations of position

For previous ruler example:

$$\Delta x = \gamma(\Delta x' + \underbrace{\beta c}_{v} \Delta t')$$

$\Delta x = \frac{L_0}{\gamma}$, $\Delta x' = L_0$

so $\frac{L_0}{\gamma} = \gamma(L_0 + \beta c \Delta t')$

$L_0\left(\frac{1}{\gamma} - \gamma\right) = \gamma \beta c \Delta t'$

$\Delta t' = \frac{L_0\left(\frac{1}{\gamma} - \gamma\right)}{\gamma \beta c}$ that's what a person in ship will say is the time diff between the 2 measurements in O

$\frac{\frac{1}{\gamma} - \gamma}{\gamma} = \frac{1}{\gamma^2} - 1 = (1-\beta^2) - 1 = -\beta^2$

so $\Delta t' = -\frac{L_0 \beta}{c}$

or $c\Delta t' = -L_0 \beta$

note $\Delta t' = t_2 - t_1$ so $\Delta t' < 0$ means t_2 is before t_1

how does time transform?

back to $\Delta x' = \gamma(\Delta x - v\Delta t)$ *

and also $\Delta x = \gamma(\Delta x' + v\Delta t')$

($v \to -v$ and swap O w O' variables)

can write $\dfrac{\Delta x}{\gamma} = \Delta x' + v\Delta t'$

or $\Delta x' = \dfrac{\Delta x}{\gamma} - v\Delta t'$

then use * $\gamma(\Delta x - v\Delta t) = \dfrac{\Delta x}{\gamma} - v\Delta t'$

use $v = \beta c \Rightarrow \beta c \Delta t' = \Delta x\left(\dfrac{1}{\gamma} - \gamma\right) + \gamma\beta c\Delta t$

$\dfrac{1}{\gamma} - \gamma = \dfrac{1-\gamma^2}{\gamma}$ and $\gamma^2 = \dfrac{1}{1-\beta^2} \Rightarrow \gamma^2 - \gamma^2\beta^2 = 1$

so $1 - \gamma^2 = -\gamma^2\beta^2$

so $\dfrac{1}{\gamma} - \gamma = -\gamma\beta^2$

so $\beta c\Delta t' = \gamma\beta c\Delta t - \gamma\beta^2\Delta x$

Lorentz transformations

$$\boxed{\begin{aligned} c\Delta t' &= \gamma(c\Delta t - \beta\Delta x) \\ \Delta x' &= \gamma(\Delta x - \beta c\Delta t) \end{aligned}}$$

$$\boxed{\begin{aligned} c\Delta t &= \gamma(c\Delta t' + \beta\Delta x') \\ \Delta x &= \gamma(\Delta x' + \beta c\Delta t') \end{aligned}}$$

writing this way puts x & ct as distances

$\Rightarrow$ space-time!

so Galilean transformation is

$$\Delta x = \Delta x' + \frac{v}{c} \cdot c\Delta t' \qquad v/c \equiv \beta$$

or $\Delta x = \Delta x' + \beta \cdot c\Delta t'$

and $c\Delta t = c\Delta t'$

Lorentz: $\Delta x = \gamma(\Delta x' + \beta c\Delta t')$

$c\Delta t = \gamma(c\Delta t' + \beta \Delta x')$

as $\beta \to 0$, $\gamma \to 1$ and Lorentz $\to$ Galilean

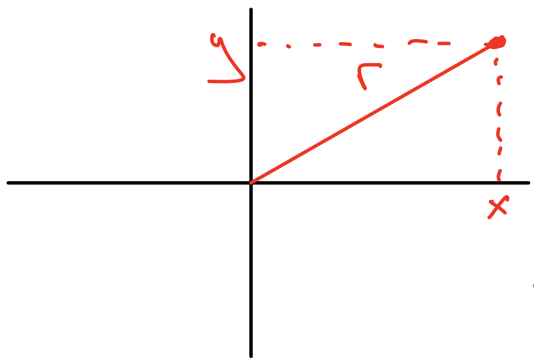
200 4/7

400 4/7

This is analogous to vector in 2D

$\Rightarrow$ Relativity is vectors in 4D

vectors & invariants



$r^2 = x^2 + y^2$

if you rotate coordinates you get a new x', y'

but $r^2 = x'^2 + y'^2$

r is "invariant" → any coordinate system will give different values for x, y but r will remain

⇒ What is the 4D invariant?

try:
$$(c\Delta t)^2 - (\Delta x)^2 = \gamma^2 (c\Delta t' + \beta \Delta x')^2 - \gamma^2 (\Delta x' + \beta c \Delta t')^2$$

$$= \gamma^2 \Big[(c\Delta t')^2 + 2\beta c \Delta t' \Delta x' + \beta^2 \Delta x'^2 - \Delta x'^2 - 2\beta c \Delta t' \Delta x' - \beta^2 c^2 \Delta t'^2 \Big]$$

$$= \gamma^2 \Big[(c\Delta t')^2 \underbrace{(1-\beta^2)}_{\frac{1}{\gamma^2}} + \Delta x'^2 \underbrace{(\beta^2 - 1)}_{-\frac{1}{\gamma^2}} \Big]$$

$$= (c\Delta t')^2 - (\Delta x')^2$$

so $(c\Delta t)^2 - (\Delta x)^2$ is invariant same in all frames

ex: take 2 "events" in frame O

event 1: at x_1, t_1

" 2: at x_2, t_2

$\Delta x = x_2 - x_1$

$\Delta t = t_2 - t_1$

then in any frame moving w/ velocity v in frame O (the O' frame)

$$(c\Delta t')^2 - (\Delta x')^2 = (c\Delta t)^2 - (\Delta x)^2$$

this is called an "invariant"

$\Rightarrow$ same value in all reference frames

(as long as they are inertial, v = constant)

Special Relativity:

relativity $\Rightarrow$ only relative velocities matter and there is no absolute velocity

special $\Rightarrow$ velocity is constant

Recap: time dilation: $\Delta t = \gamma \Delta \tau$ $\quad$ $\Delta \tau$ = time in proper frame

length contraction: $\Delta x = \frac{\Delta L_0}{\gamma}$ $\quad$ ΔL_0 = length in " "

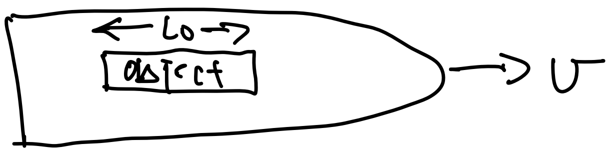
Lorentz transformation:

$$\left.\begin{aligned} x &= \gamma(x' + \beta c t') \\ ct &= \gamma(ct' + \beta x') \end{aligned}\right\} x' \to x$$

$$\left.\begin{aligned} x' &= \gamma(x - \beta c t) \\ ct' &= \gamma(ct - \beta x) \end{aligned}\right\} x \to x'$$

ex: frame O'] is moving in space ship w/ velocity v relative to frame O]

→ in O'] there is an object that is 10m long and someone in O] measures its length as it moves!



in O], makes 2 measurements of enpoints at same time in O]

$\Delta x = L$ & $\Delta t = 0$

in O'] person sees the 2 measurements in O] w/ some $\Delta x' = L_0$ & $\Delta t' \neq 0$

this is why $\Delta x \neq \Delta x'$

invariant $(c\Delta t)^2 - (\Delta x)^2 = (c\Delta t')^2 - (\Delta x')^2$

$$-L^2 = (c\Delta t')^2 - L_0^2$$

so $(c\Delta t')^2 = L_0^2 - L^2$

or $\Delta t' = \dfrac{\sqrt{L_0^2 - L^2}}{c}$

note: we know $L = L_0/\gamma$ length contraction

so $L_0^2 - L^2 = L_0^2 - \frac{L_0^2}{\gamma^2} = L_0^2 \frac{\gamma^2-1}{\gamma^2}$

$\gamma^2 = \frac{1}{1-\beta^2} \Rightarrow \gamma^2 - \gamma^2\beta^2 = 1$

$\gamma^2 - 1 = \gamma^2\beta^2$

so $L_0^2 - L^2 = L_0^2 \frac{\gamma^2-1}{\gamma^2} = L_0^2 \frac{\gamma^2\beta^2}{\gamma^2} = L_0^2\beta^2$

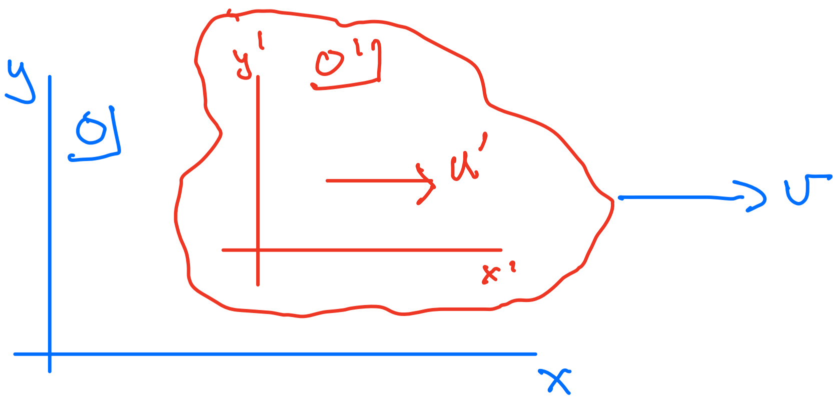
$\Delta t' = \frac{L_0\beta}{c} = \frac{L_0 v}{c^2}$

very small given v/c^2 factor!

$\Rightarrow$ can use relativistic invariant <u>always</u>
relates $\Delta x, \Delta t$ in one frame to
$\Delta x', \Delta t'$ in another

Velocity transformation

- frame O'] moves w/ vel v in frame O]
- in O'] something moves along direction of motion w/ velocity u'



ex: you are in an airplane moving w/ vel v and you throw a ball down the aisle w/ velocity u' in the airplane's frame

$\Rightarrow$ what does someone on the ground measure? call that velocity u

x' marks the position of the ball

$$x = \gamma(x' + \beta c t')$$
$$ct = \gamma(ct' + \beta x')$$

and $u = \dfrac{dx}{dt}$

$$dx = \gamma(dx' + \beta c\,dt')$$
$$c\,dt = \gamma(c\,dt' + \beta dx')$$

then $$\frac{dx}{c\,dt} = \frac{dx' + \beta c\,dt'}{c\,dt' + \beta dx'} \cdot \frac{1/dt'}{1/dt'}$$

$$= \frac{\frac{dx'}{dt} + \beta c}{c + \beta \frac{dx'}{dt}}$$

$u' = \frac{dx'}{dt}$ and $u = \frac{dx}{dt}$

so $$\frac{dx}{c\,dt} = \frac{u}{c} = \frac{u' + \beta c}{c + \beta u'}$$

or $$u = \frac{u'c + \beta c^2}{c + \beta u'} = \frac{u' + \beta c}{1 + \beta u'/c}$$

$\beta c = v$

or equivalently: $$\boxed{u = \frac{u' + v}{1 + \frac{v u'}{c^2}}}$$

check: if instead of throwing a ball w/ vel u' we shine a light, then $u' = c$

$$u = \frac{c + v}{1 + v/c} = \frac{c(1 + v/c)}{1 + v/c} = c$$ yes!

velocity of light is the same in <u>both</u> frames

ex: spaceship goes at $v = 0.8c$ in earth frame
it throws a probe forward at $u' = 0.5c$

in earth's frame $u = \dfrac{u' + v}{1 + u'v/c^2}$

$$= \left(\frac{0.5 + 0.8}{1 + 0.5 \cdot 0.8}\right) c$$

$$= 0.93c$$

ex: $u' = -c$

$$u' = \frac{-c + v}{1 - v/c} = \frac{-c(1 - v/c)}{1 - v/c} = -c \quad \checkmark$$

200 4/9

Lorentz transformation shows how space & time are combined into space-time

Space: 3 dimensions x, y, z vectors are $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

defined 4-vector (space-time vector)

components: (time, $\overrightarrow{\text{space}}$) so $(t, \vec{r})$ shorthand

$$\underset{1}{\uparrow} \qquad \underset{3}{\uparrow}$$

This is a space-time position 4-vector

⇒ But there are other 4-vectors!

Velocity 4-vector

start: take $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ and unravel:

$$\gamma^2 = \frac{1}{1-\beta^2} \Rightarrow \gamma^2 - \gamma^2\beta^2 = 1$$

$\beta = v/c$ so this is $\gamma^2c^2 - \gamma^2v^2 = c^2$

this looks like an invariant!

remember $(ct)^2 - x^2$ = same value in all frames

if we define 4-dimensional analog of velocity as:

4-d vel $\rightarrow V = (\gamma c, \gamma\vec{v})$

then the invariant is $c^2 \Rightarrow$ same in all ref frames!

now multiply by m_0, the mass of an object

$m_0 \equiv$ rest mass $\rightarrow$ mass as measured in the proper frame

"time" "space"

define 4-momentum $P = m_0 V = (m_0\gamma c, m_0\gamma\vec{v})$

4-dimensional velocity

this is the 4-D analog of momentum

invariant is $m_0^2c^2$:

$$m_0^2\gamma^2c^2 - m_0^2\gamma^2v^2 = (m_0\gamma c)^2 - (m_0\gamma\beta c)^2 = m_0^2c^2$$

multiply by c^2: $(m_o \gamma c^2)^2 - (\gamma\beta m_o c^2)^2 = (m_o c^2)^2$

let $E = m_o \gamma c^2$ relativistic energy

$p = m_o \gamma v = m_o \gamma \beta c^2$ " momentum

then the invariant is (multiply by c^2)

$$\underbrace{(m_o \gamma c^2)^2}_{E^2} - \underbrace{(m_o \gamma v)^2}_{p^2} c^2 = (m_o c^2)^2$$

so $\boxed{E^2 - p^2c^2 = (m_o c^2)^2}$

notice $m_o c^2$ is independent of velocity!

write $E = \sqrt{p^2c^2 + (m_o c^2)^2} = m_o c^2 \sqrt{1 + \left(\frac{pc}{m_o c^2}\right)^2}$

the term $\frac{pc}{m_o c^2}$ is always small except when $\beta \to 1$ $(p = \gamma m_o v)$

so expand: $(1 + x^2)^{1/2} \approx 1 + \frac{x^2}{2}$

so $E \to m_o c^2 \left(1 + \left(\frac{pc}{2 m_o c^2}\right)^2\right)$

$= m_o c^2 + \frac{p^2c^2}{2 m_o c^2}$

$= m_o c^2 + \frac{p^2}{2 m_o}$

$$\frac{p^2}{2m_0^2} = \frac{m_0^2 v^2}{2m_0^2} = \frac{1}{2}m_0 v^2 \Rightarrow \text{Kinetic energy}$$

so $E = \underbrace{m_0c^2}_{\text{rest mass energy}} + \underbrace{KE}_{\text{kinetic}}$

from above: $E = m_0\gamma c^2$ so we write $m = \gamma m_0$ to get famous formula

$$\boxed{E = mc^2}$$

400 4/9

What does this formula mean?

- when $\gamma = 1$ $(\beta = 0)$, $E = m_0c^2$

  this is called rest-energy

  All particles that have mass have an "internal" rest energy even if they are not moving (in their proper frame)

Ex: 1 gm particle at rest

$$E = mc^2 = 10^{-3}\,kg \cdot (3\times10^8\,m/s)^2$$

$$= 9\times10^{13}\,J \Leftarrow \text{enormous!}$$

note: 1 BTU = 1055 J so 1 gram contains:

$$E = 9\times10^{13}\,J * \frac{1\,BTU}{1055\,J} = 8.5\times10^{10}\,BTU$$

$= 0.085$ Trillion BTU

in 2018, state of MD used 1400 Trillion BTU

so $14\ T\ BTU * \frac{1g}{.085\,TBTU} = 165\,gm \sim \frac{1}{3}$ pound!

$\frac{1}{3}$ pound of mass contains enough mass-energy to power all of MD for a year

so mass contains <u>huge</u> amounts of energy
How to tap into it?

$\Rightarrow$ very hard to convert mass to energy
(burning does <u>not</u> do it, just releases chemical energy)

possibilities

1. nuclear energy
   Fission: $U \rightarrow Kr + Ba$ + energy
   $m(U) > m(Kr) + m(Ba)$
2. matter-antimatter
   $\bar{p} + p \Rightarrow$ energy 100% efficient
   antiproton proton

so take 165/2 = 82 grams matter +
82 " anti-matter
this is enough energy to power MD for a year
but making $\bar{p}$ is VERY expensive
$\Rightarrow$ have to make it 1 $\bar{p}$ at a time

$$82 \times 10^{-3}\, kg * \frac{1\bar{p}}{1.67 \times 10^{-27}\, kg} = 4.9 \times 10^{25}\ \bar{p} \text{ particles}$$

if you can make $10^6\ \bar{p}/sec$, need 4.9×10^{19} sec
Age of universe is $14 \times 10^9\, yr * \frac{3 \times 10^7 s}{yr} = 4.4 \times 10^{17}$ sec
100x age of universe!

The rest mass energy is the energy of something in the proper frame

- if a mass m_o moves w/velocity v in O

$E = m_o c^2$ in proper frame O'

$\beta = v/c$ & $\gamma = \frac{1}{\sqrt{1-\beta^2}}$ vel of O' in O

energy in O is given by

$$E^2 = (m_o c^2) + (pc)^2$$

where $E = \gamma m_o c^2$ & $p = \gamma m_o v$
$= \gamma m_o \beta c$

if $pc \ll m_0c^2$ then can write

$$E = m_0c^2 + KE \qquad KE = \frac{1}{2}mv^2$$

as usual

note: some physicists say $E = mc^2$

where $m = \gamma m_0$ is the mass

Then when you add energy, it speeds up and γ increases.

Does this mean mass increases?

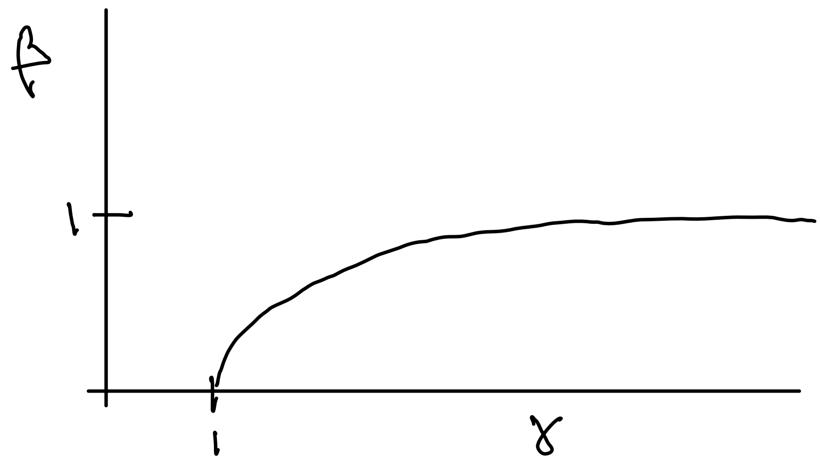
Well, it's true that $E = m_0c^2$ and if you were on O' and measured mass of particle in O' you would measure

$$m = \gamma m_0$$

But mass isn't really increasing. γ is increasing!

so as you add energy, γ increases but velocity increases slowly and will <u>never</u> get exactly to $\beta = 1$

Here is a plot of β vs γ



$p = \gamma m_0 v = \gamma \beta m_0 c$ and $E = \gamma m_0 c^2$

$E^2 = (pc)^2 + (m_0 c^2)^2$

$(\gamma m_0 c^2)^2 = (\gamma \beta m_0 c^2)^2 + (m_0 c^2)^2$

cancel out m_0^2 everywhere:

$\gamma^2 c^4 = \gamma^2 \beta^2 c^4 + c^4$

divide by c^4:

$\gamma^2 = \gamma^2 \beta^2 + 1$

$\gamma^2 (1 - \beta^2) = 1$

$$\gamma^2 = \frac{1}{1-\beta^2} \quad \checkmark$$

Recap: $\beta = v/c \Rightarrow v = \beta c$

$$\gamma = \frac{1}{\sqrt{1-\beta^2}}$$

$$\Delta x = \gamma(\Delta x' + \beta c \Delta t')$$
$$c\Delta t = \gamma(c\Delta t' + \beta \Delta x')$$

to get $\Delta x'$ & $c\Delta t'$ in terms of Δx, $c\Delta t$, let $v \to -v$ and swap primes

$$\Delta x' = \gamma(\Delta x - \beta c \Delta t)$$
$$c\Delta t' = \gamma(c\Delta t - \beta \Delta x)$$

Doing problems: use units of years for time and light-years for distance

then $c \Rightarrow$ "light" unit

ex: ship moves w/ velocity $v = \beta c$ m/s $\beta = 0.5$

goes dist in time Δt in years; $\Delta t = 5$ years

$$d = v\Delta t = \beta c \cdot \Delta t = \underset{\text{"light"}}{c} \cdot \underset{\text{years}}{\beta \Delta t}$$

$d = 0.5 \times 5 \text{ years} \times c = 2.5$ light-years

Relativistic 4-momentum $\underline{P} = (E, \vec{p})$

$E = \gamma m_0 c^2$ total energy

$\vec{p} = \gamma m_0 \vec{v}$ momentum vector

and $p = \gamma m_0 v = \gamma m_0 \beta c$ momentum value

invariant: $E^2 - (pc)^2 = (m_0 c^2)^2$

or $E^2 = (pc)^2 + (m_0 c^2)^2$

and $E = \sqrt{(pc)^2 + (m_0 c^2)^2}$

as $pc << m_0 c^2$

$$E \to m_0 c^2 + \underbrace{\frac{p^2}{2m_0}}_{\frac{1}{2} m_0 v^2 \text{ KE}}$$

ex: electron mass $= 9.109 \times 10^{-31}$ kg rest mass

$m_e = 9.109 \times 10^{-31}$ kg

$m_e c^2 = 9.109 \times 10^{-31} \text{ kg} * (3 \times 10^8 \frac{m}{s})^2$

200 4/11 $= 8.198 \times 10^{-14}$ J

remember previous chapter, eV electron volt as a measure of energy:

a charged particle w/ charge q going thru a potential change ΔV will gain ("downhill") or lose ("uphill") an amount of energy $E = q\Delta V$

if $q = +1.6\times10^{-19}C$ & $\Delta V = 1$ volt then

$$E = 1.6\times10^{-19} C \cdot 1V = 1.6\times10^{-19} J$$

we can define $\underline{1eV = 1.6\times10^{-19} J}$ new unit of E

so rest energy of electron will be

$$E_0 = m_0c^2 = 8.198\times10^{-14} J * \frac{1eV}{1.6\times10^{-19} J}$$

$$= 511\times10^3 eV = 511\ keV = 0.511\ MeV$$

$10^3 eV = 1\ keV$ thousand

$10^6 eV = 1\ MeV$ million

$10^9 eV = 1\ GeV$ billion

can write $E_0 = 0.511\ MeV$ for electron rest energy

back to $E_0 = m_0c^2$

units: $[E_0] = eV = [m]c^2$

so if E_0 is in eV then $[m] = \frac{eV}{c^2}$

this is a new unit of mass

then $p = m_0\gamma\beta c \Rightarrow$ units: $[p] = [m]c = \frac{eV}{c^2}\cdot c = \frac{eV}{c}$

so we use units for energy $[E] \Rightarrow eV$

momentum $[p] \Rightarrow \frac{eV}{c}$

mass $[m] = \frac{eV}{c^2}$

this simplifies calculations enormously

ex: election energy is 1.0MeV

find: γ, β, p

$E = \gamma m_0 c^2 = 1.0\,MeV$ and $m_0 = 0.511\,MeV/c^2$

so $\gamma = \frac{E}{m_0 c^2} = \frac{E\,(MeV)}{m_0\left(\frac{MeV}{c^2}\right)\cdot c^2} = \frac{1.0\,MeV}{0.511\,MeV} = 1.956$

$$\gamma = \frac{1}{1-\beta^2} \Rightarrow \gamma^2(1-\beta^2) = 1$$

$$\gamma^2 - \gamma^2\beta^2 = 1$$

$$\gamma^2 - 1 = \gamma^2\beta^2$$

$$\beta^2 = \frac{\gamma^2-1}{\gamma^2} \Rightarrow \beta = \sqrt{\frac{\gamma^2-1}{\gamma^2}} = \sqrt{\frac{1.956^2-1}{1.956^2}} = 0.86$$

$$p = \gamma m_0 \beta c = \gamma \frac{m_0 c^2}{c^2}\beta c = \gamma m_0 c^2 \frac{\beta}{c}$$

$$= 1.956\left(0.511\,\frac{MeV}{c^2}\right)\cdot c^2 \cdot \frac{0.86}{c} = 0.86\,MeV/c$$

or even easier: $E^2 - (pc)^2 = (m_0 c^2)^2$

so $p^2c^2 = E^2 - (mc^2)^2 = (1.0\,MeV)^2 - (0.511\,MeV)^2$

$= (0.86\,MeV)^2$

$p = 0.86\,MeV/c$ ✓

ex: election has $v = 2.5\times10^8$ m/s

$$\beta = \frac{2.5\times10^8}{3\times10^8} = 0.833$$

$$\gamma = \sqrt{\frac{1}{1-\beta^2}} = 1.81$$

$$p = m_0 v\gamma = 9.109\times10^{-31}\,\text{kg} * 2.5\times10^8\,\frac{\text{m}}{\text{s}} * 1.81$$
$$= 4.12\times10^{-22}\ \text{kg m/s}$$

$$E = m_0c^2\gamma = 8.198\times10^{-14}\,\text{J}\cdot1.81 = 1.48\times10^{-13}\,\text{J}$$

now convert to eV

$$pc = 4.12\times10^{-22} * 3\times10^8 = 1.24\times10^{-13}\,\text{J}$$
$$= 1.24\times10^{-13}\,\text{J} * \frac{1\,\text{eV}}{1.6\times10^{-19}} = 7.72\times10^5\,\text{eV}$$

$= 0.77$ MeV so $p = 0.77$ MeV/c

$$E = 1.48\times10^{-13}\,\text{J} * \frac{1\,\text{eV}}{1.6\times10^{-19}\,\text{J}} = 9.27\times10^5\,\text{eV}$$

$= 0.93$ MeV

no do the proplem in eV from start:

$$p = \gamma m_0 v = \gamma m_0 c^2 \cdot \frac{v}{c^2} = \gamma(\underbrace{m_0c^2}_{0.511\text{ MeV}})\beta/c$$

$= 1.81 * 0.511$ MeV $* 0.83/c$

$= 0.77$ MeV/c

$$E = \gamma m_0 c^2 = 1.81\times0.511\ \text{MeV} = 0.93\ \text{MeV}$$

voila!

Recap: $\boxed{E = \gamma m_0 c^2}$ energy

$= m_0 c^2 + \frac{1}{2} m_0 v^2$ as $\beta \to 0$

$= \underbrace{E_0}_{\text{rest energy}} + KE$ as $\beta \to 0$

$\boxed{p = \gamma m_0 \beta c} = \gamma m_0 v$ momentum

$$p = \frac{m_0 v}{\sqrt{1 - v^2/c^2}}$$ solve for v

write $$\frac{v}{\sqrt{1 - v^2/c^2}} = \frac{vc}{\sqrt{c^2 - v^2}} = \frac{c}{\sqrt{\frac{c^2}{v^2} - 1}}$$

so $$p = \frac{m_0 c}{\sqrt{\frac{c^2}{v^2} - 1}} \Rightarrow p^2 \left(\frac{c^2}{v^2} - 1 \right) = (m_0 c)^2$$

$$\frac{c^2}{v^2} = 1 + \left(\frac{m_0 c}{p} \right)^2$$

$$\Rightarrow \boxed{v = \frac{c}{\sqrt{1 + \left(\frac{m_0 c}{p} \right)^2}}}$$

also: $E = \gamma m_0 c^2$

$p = \gamma m_0 \beta c \Rightarrow pc = \gamma m_0 \beta c^2$

$$\therefore \frac{pc}{E} = \frac{\gamma m_0 \beta c^2}{\gamma m_0 c^2} = \beta$$

$$\boxed{\beta = \frac{pc}{E}}$$

ex:- a proton and anti-proton move towards each other at equal & opposite speed v

proton mass is $m_0 = 1.67 \times 10^{-27}$ kg
anti- " " is the same

$$m_0 c^2 = 1.67 \times 10^{-27}\,\text{kg} * \left(3 \times 10^8 \frac{m}{s}\right)^2$$
$$= 1.503 \times 10^{-10}\,\text{J} * \frac{1\,\text{eV}}{1.6 \times 10^{-19}\,\text{J}} = 0.939 \times 10^9\,\text{eV}$$
$$= 939\,\text{MeV}$$

now when p & $\bar{p}$ (anti-proton) hit, their entire energy gets turned into another particle (called the X-particle) which has a mass of

$$m_{0x} = 5.7\,\text{GeV}/c^2$$

what is the energy of the p & $\bar{p}$?

Before: $E_{tot} = E_p + E_{\bar{p}} = \gamma m_0 c^2 + \gamma m_0 c^2$
$= 2\gamma m_0 c^2$

$\vec{P}_{tot} = \vec{P}_p - \vec{P}_{\bar{p}} = 0$ opposite directions

After: $E_0 = M_{0x} c^2$ not moving

$$E_{0x} = M_{0x} c^2 = 5.7\,\text{GeV} = 2\gamma \underbrace{m_0 c^2}_{\text{proton rest mass}} = 2\, E_{0p}$$

$E_{0p} = \gamma m_0 c^2$

so $2E_{op} = E_{o\chi} = 5.7\,GeV$

$$E_{op} = \frac{5.7}{2}\,GeV = 2.85\,GeV$$

$$= \gamma m_o c^2$$

$$\gamma = \frac{E_{op}}{m_o c^2} = \frac{2.85}{0.939} = 3.04$$

velocity of protons: $\gamma = \frac{1}{\sqrt{1-\beta^2}} = 3.04$

$$1-\beta^2 = \frac{1}{(3.04)^2} = 0.108$$

$$\beta^2 = 1 - 0.108 = 0.89$$

$$\beta = 0.944$$

$$v = \beta c = 0.944 \times 3\times10^8 \frac{m}{s}$$

$$= 2.8\times10^8 \frac{m}{s}$$

200 4/14

400 4/14

ex: 2 protons w/ equal & opposite velocity collide when they come to rest, energy is converted to a pion w/ $E_0 = m_0c^2 = 135\,MeV$
what is initial proton velocity?

BEFORE: $E = 2E_p = 2\gamma E_{op}$

FINAL : $E = 2E_{op} + E_{o\pi}$

note: momentum $= 0$ BEFORE <u>and</u> AFTER so $2\gamma m_{op} c^2 = 2 m_{op} c^2 + m_{o\pi} c^2$

$2\gamma m_p = 2 m_p + m_\pi$

$$\gamma = \frac{2m_p + m_\pi}{2m_p} = 1 + \frac{m_\pi}{2m_p}$$

$$= 1 + \frac{135}{2 \cdot 939} \simeq 1.072$$

$$\gamma^2 = \frac{1}{1-\beta^2} \qquad (1-\beta^2)\gamma^2 = 1$$

$$\beta^2\gamma^2 = \gamma^2 - 1$$

$$\beta^2 = \frac{\gamma^2-1}{\gamma^2}$$

so $\beta = \sqrt{\dfrac{(1.072)^2 - 1}{(1.072)^2}} \simeq 0.36$

$v = \beta c = 0.36 \times 3 \times 10^8 \frac{m}{s} = 1.08 \times 10^8 \frac{m}{s}$

ex: electron is given 1MeV of kinetic energy
$\Rightarrow$ what is γ & β?

non relativistically: $KE = \frac{1}{2} m_0 v^2 = \frac{1}{2} m_0 c^2 \beta^2$

relativistically, $E = \gamma m_0 c^2$

we saw that $E \xrightarrow[\beta \to 0]{} m_0 c^2 + \frac{1}{2} m v^2 = m_0 c^2 + KE$

and $E = \gamma m_0 c^2$

so if we take $E - m_o c^2 \xrightarrow[\beta \to 0]{} \frac{1}{2} m v^2 = KE$

so $E - m_o c^2 = (\gamma - 1) m_o c^2$ is the relativistic KE!

so $(\gamma - 1) m_o c^2 = 1\,MeV$ in this problem

$(\gamma - 1)(0.511\,MeV) = 1\,MeV$

$\gamma - 1 = \frac{1}{0.511} = 1.956$

$\gamma = 2.956$

$\beta = \sqrt{1 - \frac{1}{\gamma^2}} = 0.94 \Rightarrow v = \beta c$

$= 0.94 \times 3 \times 10^8\ m/s$

$= 2.8 \times 10^8\ m/s$